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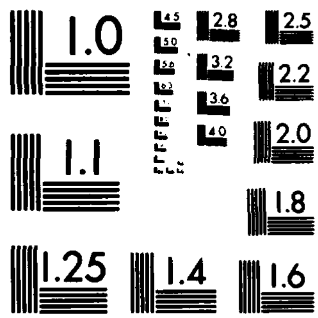

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STATISTICAL INFERENCE FOR RELIABILITY FROM STRESS STRENGTH RELATIONSHIPS: THE NORMAL CASE

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This paper examines statistical inference for $P(Y < X)$, where X and Y are independent normal variates with unknown means and variances. The case of unequal variances is stressed. X can be interpreted as the strength of a component subjected to a stress Y, and $P(Y < X)$ is the components reliability. For point estimation, a predictive estimator which can be calculated from the Behrens-Fisher distribution is derived and compared with the maximum likelihood and uniformly minimum variance unbiased estimators through a simulation study. Two approximate methods for obtaining confidence intervals and an approximate Bayesian probability interval is obtained. The actual coverage probabilities of these intervals is examined by simulation.

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Key Words: Reliability, predictive, maximum likelihood, UMVUE estimators, Bayesian and sampling theory, interval estimators

Work Unit Number 4 (Statistics and Probability)

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SIGNIFICANCE AND EXPLANATION

The reliability, say R, of a system is often of interest, where $R = P(Y < X)$. Here, the random variables X and Y are assumed to be independent, with X the strength of a component of interest subjected to a stress Y. Our results are for the case where X and Y are normally distributed with unknown means and unknown variances. (This of course means that after transformation to logs, that our results apply for the case where X and Y are distributed as log normal). For the case of point estimation of R, we obtain a new estimator based on the predictive distribution of Y - X. This is compared, via a simulation study, with 2 estimators mentioned in the literature without too much analysis, namely the "nearly" maximum likelihood estimate and the Rao-Blackwell UMVUE. The simulation study supplies evidence for the conventional wisdom that the use of the "nearly" maximum likelihood esimtate is well advised.

In addition, we obtain for the first time two approximate methods for constructing confidence intervals for R, as well as an approximate Bayesian probability interval. The actual coverage probabilities of these intervals is examined by simulation.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

STATISTICAL INFERENCE FOR RELIABILITY FROM STRESS STRENGTH RELATIONSHIPS: THE NORMAL CASE

Benjamin Reiser* and Irwin Guttman

1. INTRODUCTION.

Let X and Y be independent normal random variables with means and variances μ_x, σ_x^2 and μ_y, σ_y^2 respectively. Interest is focused on statistical inference for the parameter

$$R = P(Y < X) = \Phi\left(\frac{\mu_x - \mu_y}{\sqrt{\sigma_x^2 + \sigma_y^2}}\right)$$

where Φ is the standard normal cumulative distribution function. If X is interpreted as the strength of a component subjected to a stress Y, the resulting reliability is given by R. This type of situation is of particular interest in probabilistic mechanical design (for example, see Haugen (1980)). We further assume that, random samples of size n and m, say $\underset{\sim}{X} = (X_1, \ldots, X_n)'$ and $\underset{\sim}{Y} = (Y_1, \ldots, Y_m)'$ are available on X and Y. In this situation point and interval estimation procedures are discussed.

Related problems have been widely presented in the literature. Birnbaum (1956), Birnbaum and McCarty (1958) and Owen et al. (1964) present non-parametric confidence limits for this problem. In addition Owen et al. (1964) discuss the normal case for $m = n$ and $\sigma_x = \sigma_y$ and for paired observations on the jointly normal variates X and Y. Enis and Geisser (1971) look at various situations from a Bayesian viewpoint giving predictive estimates and posterior Bayesian limits but do not discuss the case where σ_x is not

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necessarily equal to σ_y. Procedures if the parameters μ_y, σ_y are known are given in Church and Harris (1970) and Mazumdar (1970).

For the particular problem we have defined, Lloyd (1980) and Haugen (1980) use the (almost) maximum likelihood estimate

$$\hat{R}_1 = \Phi\left(\frac{\bar{X} - \bar{Y}}{\sqrt{S_x^2 + S_y^2}}\right) \tag{1.1}$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$, $\bar{Y} = \sum_{i=1}^{m} Y_i/m$, $(n - 1)S_x^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2$, and $(m - 1)S_y^2 = \sum_{i=1}^{m} (Y_i - \bar{Y})^2$ while Downton (1973) derives the uniformly minimum variance unbiased estimator (UMVUE) of R, which we will denote by $\hat{R}_2$. (For completeness, we state the formula for $\hat{R}_2$ in Appendix AI.) Lloyd obtains, by the usual propagation of errors method, an estimate of the variance of $\hat{R}_1$, which with the assumption that $\hat{R}_1$ is normally distributed, then gives approximate confidence limits. Since $0 < R_1 < 1$ with R_1 usually close to one this procedure can give misleading limits. Haugen (1980) obtains confidence limits of $\mu_x - \mu_y$ based on the standard approximate t solution to the Behrens-Fisher problem. The limits for $\mu_x - \mu_y$ are simply divided by $\sqrt{S_x^2 + S_y^2}$ and in turn Φ of these values are calculated providing limits on R. The properties of this procedure are not well understood due to the division by $a = \sqrt{S_x^2 + S_y^2}$, without accounting for the variation of a. Kececioglu and Lamarre (1978) attempt to generalize the standard procedure for deriving confidence limits for the tail probability of one normal distribution to this case. Unfortunately their derivation involves a fundamental error which will be discussed in more detail below.

In Section 2 of this paper, we derive a point estimator of R using a Bayesian predictive approach, whose characteristics are compared with those of $\hat{R}_1$ and $\hat{R}_2$ in Section 3, by means of a simulation study. In Section 4, we derive confidence limits on R, first from the frequentist approach involving a non-central t distribution, obtained by approximating the distribution of a weighted sum of independent Chi-squares by the distribution of a scaled χ^2- variable by equating the first two moments. A well known approximation to the non-central t is then utilized to drive a different confidence interval. Finally, we use a Bayesian approach to derive a (posterior) confidence interval for R. These confidence intervals are compared in Section 4.4 by means of a further simulation study.

2. PREDICTIVE APPROACH

There is another approach which will generate a point estimate of the reliability functions, namely the Bayesian approach through the relevant predictive distribution. So suppose in general, that the random variable w, whose distribution is $f(w|\underset{\sim}{\theta})$, with $\underset{\sim}{\theta}$ a $(t \times 1)$ vector of parameters, is to be observed independently of the data at hand, say $w_1,\ldots,w_n$, where the w_i are independently and identically distributed observations on w. Then the predictive distribution of w, say h, given the data $\underset{\sim}{w} = (w_1,\ldots w_n)'$, is defined as

$$h(w|\underset{\sim}{w}) = \int_{\underset{\sim}{\theta}} h(w|\underset{\sim}{\theta})p(\underset{\sim}{\theta}|\underset{\sim}{w})d\underset{\sim}{\theta} \ , \tag{2.1}$$

where $p(\underset{\sim}{\theta}|\underset{\sim}{w})$ is the posterior distribution of $\underset{\sim}{\theta}$, given $\underset{\sim}{w}$, that is,

$$p(\underset{\sim}{\theta}|\underset{\sim}{w}) = K[\prod_{i=1}^{n} f(w_i|\underset{\sim}{\theta})]p(\underset{\sim}{\theta}) \ , \tag{2.1a}$$

with $p(\underset{\sim}{\theta})$ the prior of $\underset{\sim}{\theta}$.

Now we are interested in the reliability R, where

$$R = P(Y < X | \mu_y, \mu_x, \sigma_x^2, \sigma_y^2) \tag{2.2}$$

and suppose we have available the data $\underset{\sim}{x} = (x_1, \ldots, x_n)'$ and $\underset{\sim}{y} = y_1, \ldots, y_m)'$, which are, respectively n and m independent observations on X, the strength variable, and Y, the stress vairable. If subsequent to the taking of $\underset{\sim}{x}$ and $\underset{\sim}{y}$, we will observe, independently, X and Y, where X has the same common distribution of the x_i and y has the same common distribution of the y_j, then, as is well known, if $x_i \sim N(\mu_x, \sigma_x^2)$, $y_j \sim N(\mu_y, \sigma_y^2)$, the application of (2.1) when the prior distribution chosen is the Jeffrey's diffuse prior

$$P(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2) \propto (\sigma_x^2 \sigma_y^2)^{-1} \tag{2.3}$$

leads to *independent predictive distributions* h_x and h_y for X and Y which are such that

$$X \sim \bar{x} + \sqrt{\left(1 + \frac{1}{n}\right) s_x^2}\; t_{n-1} \tag{2.3a}$$

and

$$Y \sim \bar{y} + \sqrt{\left(1 + \frac{1}{m}\right) s_y^2}\; t_{m-1} \tag{2.3b}$$

where t_{n-1}, t_{m-1} are independent random variables with student t-distributions of $n - 1$, $m - 1$ degrees of freedom respectively. We note that in this particular case, the above predictive distributions also have a confidence interpretation.

We now consider the predictive estimate of R defined in (2.2), namely

$$\hat{R}_3 = P(Y < X | \underset{\sim}{x}, \underset{\sim}{y}) \tag{2.4}$$

From (2.3) above, we have that (2.4) may be written as

$$\hat{R}_3 = P\left(\sqrt{(1 + \tfrac{1}{m})s_y^2}\, t_{m-1} - \sqrt{(1 + \tfrac{1}{n})s_x^2}\, t_{n-1} < \bar{x} - \bar{y} \,|\, \underset{\sim}{x}, \underset{\sim}{y}\right) \tag{2.5}$$

$$= P\left((\cos\theta)t_{m-1} - (\sin\theta)t_{n-1} < \frac{\bar{x} - \bar{y}}{\sqrt{(1 + \tfrac{1}{m})s_y^2 + (1 + \tfrac{1}{n})s_x^2}}\right)$$

$$= P\left(BFT(m - 1, n - 1) < \frac{\bar{x} - \bar{y}}{\sqrt{(1 + \tfrac{1}{m})s_y^2 + (1 + \tfrac{1}{n})s_x^2}}\right)$$

where BFT is the Behrens-Fisher t variable, degrees of freedom $(m - 1, n - 1)$, and θ given by

$$\theta = \tan^{-1} \frac{\sqrt{(1 + \tfrac{1}{n})s_x^2}}{\sqrt{1 + \tfrac{1}{m}\, s_y^2}} \tag{2.5a}$$

For a given set of data, tables of the Behrens-Fisher t are available for various values of the pairs of degrees of freedom, and 'angle' θ. These prove to be not very helpful in practice - only percentage points are available, and this coupled with the fact that the calculated value of θ (with probability one) is never a value used in the table, requires reverse double extrapolation, etc. There is, however, an approximation due to Patil (1965), who sets

$$BFT(m - 1, n - 1) = t_f/h \tag{2.6}$$

where t_f is a Student-t variable with f degrees of freedom, with

$$f = 4 + \frac{\left[\frac{m-1}{m-3}\cos^2\theta + \frac{n-1}{n-3}\sin^2\theta\right]^2}{\left[\frac{(m-1)^2}{(m-3)^2(m-5)}\cos^4\theta + \frac{(n-1)^2}{(n-3)^2(n-5)}\sin^4\theta\right]} \tag{2.6a}$$

and

$$h^2 = \frac{f}{f-2}\left[\frac{m-1}{m-3}\cos^2\theta + \frac{n-1}{n-3}\sin^2\theta\right]^{-1} \tag{2.6b}$$

This approximation seems to work well for (n,m) as small as 8, and in our simulations below we use (2.6) above in evaluating $\hat{R}_3$ of (2.5). The behaviour of the $\hat{R}_i$'s is now discussed in the following sections.

3. SIMULATION STUDY OF THE BEHAVIOUR OF THE $\hat{R}_i$.

For various sample sizes (n,m), we generated random samples on X and Y for the case where R defined in (2.2) takes on the values .99, .90 and .67. For each of the chosen (n,m), we iterated 5,000 times, and from the results computed the bias and the root mean square error (RMSE), for the $\hat{R}_i$'s. The results are given in Table 1.

The simulation results confirm that the UMVUE estimator $\hat{R}_2$ is indeed unbiased. In terms of bias, the predictive estimator $\hat{R}_3$ is, in all our calculations, worse than either $\hat{R}_1$ or $\hat{R}_2$. The bias of $\hat{R}_1$ seems not large (discrepancies are in the third place).

Referring to RMSE, none of the $\hat{R}_i$ turn out to be best uniformly. The predictive is best only for the case $R = .67$, and is otherwise the worst. We note that for $R = .67$, that $\hat{R}_1$ is second best with $\hat{R}_2$ trailing the other two, but that differences are small.

For the cases $R = .99$ or .90, $\hat{R}_3$ in terms of RMSE, is always ranked last, with $\hat{R}_1$ and $\hat{R}_2$ alternating as the best, but again with only small differences. However, the difference of $\hat{R}_1$ or $\hat{R}_2$ with $\hat{R}_3$ could be substantial. For example, for $R = .99$, $(n,m) = (10,10)$, the RMSE of $\hat{R}_1$ and $\hat{R}_2$ are .0173 and .0146 respectively, while the RMSE of $\hat{R}_3$ is .0266.

We note that as the sample sizes (n,m) increase, the behaviour of the $\hat{R}_i$ tend to be similar, and this is repeated as R tends to .5. As a

			$\hat{R}_1$		$\hat{R}_2$		$\hat{R}_3$	
R	n	m	BIAS	RMSE	BIAS	RMSE	BIAS	RMSE
	10	10	-.0047	.0173	.0000	.0146	-.0161	.0266
	10	15	-.0038	.0148	.0001	.0126	-.0129	.0221
	10	20	-.0037	.0144	-.0040	.0121	-.0119	.0209
	15	10	-.0038	.0148	.0001	.0126	-.0129	.0221
.99	15	15	-.0031	.0131	-.0001	.0114	-.0100	.0183
	15	20	-.0028	.0120	-.0041	.0104	-.0086	.0164
	20	10	-.0033	.0140	.0003	.0121	-.0115	.0204
	20	15	-.0028	.0120	.0000	.0104	-.0087	.0164
	20	20	-.0023	.0107	-.0037	.0091	-.0071	.0141
	10	10	-.0053	.0662	.0019	.0673	-.0240	.0700
	10	15	-.0052	.0605	.0007	.0613	-.0207	.0636
	10	20	-.0055	.0572	-.0009	.0576	-.0195	.0601
	15	10	-.0052	.0605	.0007	.0613	-.0207	.0636
.90	15	15	-.0036	.0533	.0011	.0539	-.0161	.0556
	15	20	-.0049	.0505	-.0025	.0510	-.0159	.0527
	20	10	-.0040	.0565	.0018	.0567	-.0180	.0590
	20	15	-.0052	.0513	.0002	.0511	-.0162	.0535
	20	20	-.0037	.0471	-.0009	.0476	-.0131	.0488
	10	10	-.0039	.1161	-.0016	.1181	-.0135	.1105
	10	15	-.0015	.1071	.0003	.1085	-.0096	.1025
	10	20	-.0013	.1007	.0000	.1018	-.0085	.0969
	15	10	-.0015	.1071	.0003	.1085	-.0096	.1026
.67	15	15	-.0011	.0946	.0003	.0957	-.0078	.0914
	15	20	-.0006	.0879	-.0014	.0882	-.0064	.0853
	20	10	-.0028	.1026	-.0003	.1044	-.0099	.0989
	20	15	.0006	.0879	.0033	.0903	-.0053	.0852
	20	20	-.0023	.0833	-.0034	.0855	-.0073	.0812

Table 1. Comparisons of the $\hat{R}_i$ based on 5,000 iterations.

$\hat{R}_1$, see (1.2); $\hat{R}_2$, see (AI.3); $\hat{R}_3$, see (2.5) consequence of this and the results above, and because of the simplicity in computing $\hat{R}_1$, we favour its use as a point estimator.

We also note that for small samples, the RMSE is quite large, so that point estimates here are not very satisfactory. It is for this reason that we now consider the question of interval estimation.

4. CONFIDENCE LIMITS ON R.

4.1 The Frequentist Approach

Usually a lower bound on the reliability is of interest and we will restrict ourselves to this case. Since Φ is a monotonically increasing function of $\delta = (\mu_x - \mu_y)/\sqrt{\sigma_x^2 + \sigma_y^2}$ finding a lower confidence bound for R is equivalent to finding one for δ. Our problem has essentially the same structure as the Behrens-Fisher problem and the solution presented below is similar in spirit to the standard approximate "t" solution to the Behrens-Fisher problem.

Note that $\bar{x} - \bar{y} \sim N\left(\mu_x - \mu_y, \frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}\right)$,

and

$$(n - 1)s_x^2/\sigma_x^2 \sim \chi_{n-1}^2$$

$$(m - 1)s_y^2/\sigma_y^2 \sim \chi_{m-1}^2 ,$$

with all three of the above being pairwise independent. Let $N = \dfrac{\sigma_x^2 + \sigma_y^2}{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}$.

Then

$$\frac{\bar{x} - \bar{y}}{\sqrt{\frac{\sigma_x^2 + \sigma_y^2}{N}}} \sim N\left(\frac{\mu_x - \mu_y}{\sqrt{\frac{\sigma_x^2 + \sigma_y^2}{N}}}, 1\right) \text{ or } N(\sqrt{N}\,\delta, 1) . \tag{4.1}$$

In the equal sample size case, $N = n = m$. In addition

$$s_x^2 + s_y^2 \sim \frac{\sigma_x^2}{n-1} \chi^2_{n-1} + \frac{\sigma_y^2}{m-1} \chi^2_{m-1}$$

or approximately

$$\underset{\sim}{a} \frac{\sigma_x^2 + \sigma_y^2}{f} \chi_f^2 \tag{4.2}$$

(The notation $\underset{\sim}{a}$ stands for "approximately distributed as") where

$$f = (\sigma_x^2 + \sigma_y^2)^2 \Big/ \frac{\sigma_x^4}{n-1} + \frac{\sigma_y^4}{m-1}$$

and $s_x^2 + s_y^2$ is distributed independently of (4.1). Defining $\hat{\delta} = \dfrac{\bar{X} - \bar{Y}}{\sqrt{s_x^2 + s_y^2}}$

we obtain from (4.1) and (4.2) that

$$\sqrt{N}\,\hat{\delta} \underset{\sim}{a} t_f(\sqrt{N}\,\delta) \tag{4.3}$$

where $t_f(\sqrt{N}\,\delta)$ denotes a non-central t distributed variate with f degrees of freedom and non-centrality parameter $\sqrt{N}\,\delta$. If $n = m$ and it is known that $\sigma_x = \sigma_y$, (4.3) holds exactly and leads to the well known sampling theory solution (see Owen et al. (1964), Enis and Geisser (1971)). By using s_x^2 and s_y^2 in the formula for N and f the estimates $\hat{N}$ and $\hat{f}$ are obtained giving the further approximation

$$\sqrt{\hat{N}}\,\hat{\delta} \underset{\sim}{a} t_{\hat{f}}(\sqrt{\hat{N}}\,\delta) \tag{4.4}$$

For equal sample sizes $N = n = m$ does not require estimation. Since the non-central t distribution has the monotone likelihood ratio property an approximate $1 - \alpha$ level lower confidence bound for δ can be obtained by solving

$$\text{Prob}(t_{\hat{f}}(\sqrt{\hat{N}}\,\delta) \le \sqrt{\hat{N}}\,\hat{\delta}) = 1 - \alpha \tag{4.5}$$

numerically for δ (Lehmann (1958)). Denoting this solution by $\delta_{1\ell}$, the $1 - \alpha$ level approximate lower confidence bound for R is then simply

$R_{1\ell} = \Phi(\delta_{1\ell})$. Kececiogle and Lamarre (1978) take a similar approach but make the mistaken assumption that $N = f + 1$.

A simplification is possible by using the well known approximation

$$(t_f(\gamma) - \gamma)/\left(1 + \frac{t(\gamma)^2}{2f}\right)^{1/2} \overset{a}{\sim} N(0,1) \ .$$

Applying this to (4.3) gives

$$\sqrt{\hat{N}}\ (\hat{\delta} - \delta)/\left(1 + \frac{\hat{N}\hat{\delta}^2}{2\hat{f}}\right)^{1/2} \overset{a}{\sim} N(0,1) \ . \tag{4.6}$$

Thus a $(1 - \alpha)$ level approximate lower confidence bound $\delta_{2\ell}$ can be obtained as

$$\delta_{2\ell} = \hat{\delta} - \left(\frac{1}{\hat{N}} + \frac{\hat{\delta}^2}{2\hat{f}}\right)^{1/2} z_{1-\alpha} \ , \tag{4.7}$$

where $z_{1-\alpha}$ is the $1 - \alpha$ standard normal percentile point. The corresponding bound on R is then $R_{2\ell} = \Phi(\delta_{2\ell})$. This result can alternatively be obtained by considering $\hat{\delta}$ asymptotically to be normally distributed with mean δ and variance estimated by the usual propagation of errors method. Details are omitted for the sake of brevity. It seems more reasonable to base inference on a normal approximation to $\hat{\delta}$ than on the normal approximation to $\hat{R}$ as suggested by Lloyd (1980) since $\hat{\delta}$ is unbounded.

Church and Harris (1970) point out that in certain cases it is possible to assume that the parameters of the stress distribution are effectively known. This is equivalent to taking $m \to \infty$. It can readily be verified that $\lim_{m\to\infty} \delta_{2\ell}$ gives the solution presented by Church and Harris.

4.2 The Bayesian Approach

The Bayesian approach also leads to bounds on the reliability. Given the data, and assuming that the usual vague priors are appropriate, viz

$$p(u_x, u_y, \delta_x^2, \delta_y^2) \propto 1/\sigma_x^2 \delta_y^2 \;, \tag{4.8}$$

then the joint posterior of $\mu_x, \mu_y, \sigma_x^2, \delta_y^2$ can be obtained. Using the joint posterior, we can in principle proceed to find the marginal posterior of

$$\delta = \frac{\mu_x - \mu_y}{\sqrt{\sigma_x^2 + \delta_y^2}} \tag{4.9}$$

and thus of $R = \Phi(\delta)$. To do this requires a triple integration. A method of doing this in closed from eludes us, but in principle we may do this numerically. We would like to indicate a simpler solution, and we first note that given σ_x^2, σ_y^2, that the posterior of γ is such that

$$\delta \sim N\left(\frac{\bar{x} - \bar{y}}{\sqrt{\sigma_x^2 + \sigma_y^2}}, \frac{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}{\sigma_x^2 + \sigma_y^2}\right) \;, \tag{4.10}$$

while the posterior of the variances are such that

$$\sigma_x^2 \sim (n - 1)S_x^2/\chi_{n-1}^2, \quad \text{independent of} \quad \sigma_y^2 \sim (m - 1)S_y^2/\chi_{m-1}^2 \tag{4.11}$$

We see now that if $n = m$, the distributional variance of δ is simply $1/n$, and to find the unconditional distribution of δ, we need only integrate (4.7) with respect to the distribution of $\sigma_x^2 + \sigma_y^2$, where the distribution of the variances σ_x^2 and σ_y^2 is given in (4.11) with $m = n$. At this point we will approximate the distribution of

$$\sigma_x^2 + \sigma_y^2 \sim \frac{a_n}{{}_1\chi^2_{n-1}} + \frac{b_n}{{}_2\chi^2_{n-1}} \tag{4.12}$$

(the ${}_i\chi^2_{n-1}$ are independent), where $a_n = (n - 1)s_x^2$, $b_n = (n - 1)s_y^2$, by letting

$$\sigma_x^2 + \sigma_y^2 \sim \frac{a}{\chi_b^2} , \tag{4.13}$$

so that by equating moments, we let

$$\begin{aligned} a &= (e_n^{-1} + 2)c_n \\ b &= (e_n^{-1} + 4) , \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} e_n &= (n - 5)^{-1}dn^2 + (1 - d_n)^2 \\ d_n &= a_n/(a_n + b_n) \\ c_n &= (n - 3)^{-1}(a_n + b_n) . \end{aligned} \tag{4.15}$$

If we set

$$w = \sigma_x^2 + \sigma_y^2 \tag{4.16}$$

then w has the approximate distribution

$$f(w|\underset{\sim}{x};\underset{\sim}{y}) = \frac{a^{b/2}}{2^{b/2}\Gamma\left(\frac{b}{2}\right)} w^{-\left(\frac{b}{2} + 1\right)} \exp(-a/2w) \tag{4.17}$$

Combining (4.10) with (4.17) and integrating out w, we find that the unconditional posterior of δ is such that

$$p(\delta|\underset{\sim}{x},\underset{\sim}{y}) \propto \int_0^\infty \exp\left\{- \frac{n}{2} \left[\delta - \frac{\bar{x} - \bar{y}}{\sqrt{w}}\right]^2 \right\} w^{-\left(\frac{b}{2} + 1\right)} \exp(-a/2w)dw \tag{4.18}$$

or, after setting $u = w^{-1/2}$ $(|dw/du| = 2u^{-3})$

$$p(\delta|\underset{\sim}{x},\underset{\sim}{y}) = K\int_0^\infty u^{b-1}\exp[-\frac{1}{2}\{n[\delta - (\bar{x} - \bar{y})u]^2 + au^2\}]du \qquad (4.19)$$

Inegrating both sides of (4.19) will respect to δ yields (a proof is given in Appendix II)

$$K = \sqrt{n}\ a^{b/2}/\sqrt{\pi}\ 2^{(b-1)/2}\Gamma(b/2) \qquad (4.20)$$

To find a lower $(1 - \alpha)$ Bayesian bound for δ, say $\delta_{3\ell}$ we need to solve, for $\delta_{3\ell}$, the equation

$$1 - \alpha = \int_{\delta_{3\ell}}^\infty p(\delta|\underset{\sim}{x},\underset{\sim}{y})d\delta \qquad (4.21)$$

which after an interchange of the order of integration, may be written as

$$1-\alpha = \frac{a^{b/2}}{2^{\frac{b-2}{2}}\Gamma(\frac{b}{2})}\int_0^\infty u^{b-1}\exp\{\frac{-au^2}{2}\}\{\int_{\delta_{3\ell}}^\infty \frac{\sqrt{n}}{\sqrt{2\pi}}\exp - \frac{n}{2}[\delta - (\bar{x}-\bar{y})u]^2 d\delta\}du \qquad (4.22)$$

so that we have

$$1 - \alpha = \frac{a^{b/2}}{2^{\frac{b-2}{2}}\Gamma(\frac{b}{2})}\int_0^\infty u^{b-1}\exp\{\frac{-au^2}{2}\}\Phi[\sqrt{n}((\bar{x} - \bar{y})u - \delta_{3\ell})]du , \qquad (4.23)$$

and we see that it is necessary to solve (4.23) numerically for $\delta_{3\ell}$. Once found we have a lower Bayessian bound for R, viz

$$R_{3\ell} = \Phi(\delta_{3\ell}) . \qquad (4.24)$$

Using (4.18)-(4.20), it can be shown that for large n,

$$E(\delta|\underset{\sim}{x},\underset{\sim}{y}) \approx \hat{\delta}, \quad Var(\delta|\underset{\sim}{x},\underset{\sim}{y}) \approx (\frac{1}{\hat{N}} + \frac{\hat{\delta}^2}{2\hat{f}}) , \qquad (4.25)$$

and we note the similarity to the sampling theory results of (4.6). The question of the asymptotic normality of δ, given $\underset{\sim}{x}$, $\underset{\sim}{y}$, remains open.

4.3 An Example based on Data of Kececioglu and Lamarre

Kececioglu and Lamarre (1978) give data pertaining to a mechanical component (their Example 3) that yields

$$\begin{array}{ll} \bar{x} = 170{,}000 \text{ psi}, \quad S_x = 5{,}000 \text{ psi}, & \\ & n = m = 32 \\ \bar{y} = 144{,}500 \text{ psi}, \quad S_y = 8{,}900 \text{ psi}, & \end{array}$$

For a 90% confidence lower bound, we have on the basis of the above data,

$$R_{1\ell} = .9822, \quad R_{2\ell} = .9818, \quad R_{3\ell} = .9926$$

These were of course calculated as indicated in Section 4.1 and 4.2. It is interesting to note that Kececioglu and Lamarre obtain the bound of .980 for this set of data, for $1 - \alpha = .90$.

4.4 Simulation Study of $R_{i\ell}$

The first part of our simulation study compares the coverage properties of the $R_{i\ell}$'s. We used 2,500 iterations with $n = m = 10$, 15 and 20, with $R = .99$, and calcualted the $R_{i\ell}$ for $1 - \alpha = .90$. We then counted the number of times that R exceeded the calculated $R_{i\ell}$, with the proportions out of 2500 tabulated in Table 2. This procedure was carried out for σ_y^2/σ_x^2 set equal to 1 and 3. We note that the non-Bayesian methods have coverage very close to the nominal confidence level of .90, even for very small sample sizes. We remark that we have examined these bounds for many cases and have observed similar behavior, that is, the bounds $R_{1\ell}$ and $R_{2\ell}$ obtained are strikingly similar, so that, as expected, the corresponding estimated confidence levels are quite close. This implies that $R_{2\ell}$, derived as an approximation to $R_{1\ell}$, is a quite good approximation, and due to the ease of computation of $R_{2\ell}$, its use is thus recommended.

Table 2. Estimated Confidence levels for $1 - \alpha = .9$ $(n = m)$ with $R = .99$, based on 2,500 iterations.

σ_y^2/σ_x^2	$n = m$	$R_{1\ell}$	$R_{2\ell}$	$R_{3\ell}$
	10	.908	.897	.956
1	15	.903	.894	.940
	20	.912	.899	.939
	10	.899	.888	.940
3	15	.899	.889	.928
	20	.894	.889	.921

The coverage of the Bayesian method is farther from the nominal than the confidence methods but this improves with larger sample size corresponding to the results mentioned at the end of Section 4.

The second part of our simulation study examines $R_{1\ell}$ and $R_{2\ell}$ for unequal sample sizes and is presented in Table 3. Again we see that the coverage is quite close to the nominal confidence level of .90 and that the simple $R_{2\ell}$ method performs satisfactorily. Since for equal sample sizes just f needs to be estimated while in the unequal case both f and N are estimated we had expected our methods to perform not as well in the unequal case as the equal case. Comparing Tables 1 and 3 we see that this is incorrect and that in fact there is no deterioration for unequal sample sizes.

Table 3. Estimated Confidence levels for $1 - \alpha = .9$

$(n = m)$, $R = .99$, based on 2,500 iterations

σ_y^2/σ_x^2	(n,m)	10,15	10,20	15,10	15,20	20,10	20,15
1	$R_{1\ell}$	.900	.900	.900	.901	.901	.904
	$R_{2\ell}$	.894	.892	.889	.891	.890	.899
3	$R_{1\ell}$	.907	.911	.892	.901	.885	.892
	$R_{2\ell}$	.898	.901	.882	.891	.875	.884

5. DISCUSSION

The above computations of lower confidence bounds can easily be extended to the case where strength and stress are sums of independent normal variates, each of which can be sampled separately. In addition if X and Y are lognormally distributed with $X,Y > 0$, and since $P(Y < X) = P(\log Y < \log X)$, all the results of this paper can be applied after taking the log transformation of the observations.

In many situations X and Y are not observed directly but one or both of them are computed as functions of random variables which can be observed (see Haugen (1980) for example). Frequently these functions are multiplicative in nature (Avakov (1983)) and the use of the lognormal is reasonable. If these multiplicative factors can be assumed to be lognormally distributed, then with the availability of sampling data confidence bounds can be calculated in addition to the point estimates of Avakov.

Although the normal distribution is most commonly assumed for both strenth and stress there are often theoretical or empirical justifications for the use of other distributions. Distributions such as the lognormal, exponential, Gamma, Weiball, extreme value, and Maxwellian have been suggested (Haugen (1980), Dhillon (1980)). Note that the strength and stress distributions may come from a different family. Although there has been some discussion in the literature on point estimation in the non-normal case (see for example Beg (1980) and the references cited there) very few results are available on interval estimation. Basu (1981) gives some results in the Gamma case and we have shown above how the lognormal situation can be handled. Unfortunately, inference will be highly sensitive to parametric assumptions and where only small sample sizes are available it will be very difficult to decide which parametric form is appropriate. The nonparametric approach is not very helpful here due to its extremely conservative nature (Basu (1981)). There seems to be a need for some semi-nonparametric approach which would not apply to all distributions but just to some set of "reasonable" ones.

APPENDIX AI

As mentioned in the Introduction section, we state in this Appendix, the formula for $\hat{R}_2$, the uniformly minimum variance unbiased estimator (UMVUE) of $R = P(X < Y)$, as derived by Downton (1973), where $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$, with X and Y independent. We are assuming that samples $X_1, \ldots, X_n$ of n independent observations on X and, independently, $Y_1, \ldots, Y_m$, m independent observations on Y, are available. Using the notation of our paper, Downton shows that $\hat{R}_2$ is given as follows: Define

$$A = \frac{\bar{x} - \bar{y}}{S_x(n-1)n^{-1/2} + S_y(m-1)m^{-1/2}} \tag{AI.1}$$

and for $|v| < 1$, let

$$h(v) = \begin{cases} 0 & \text{if} \quad \phi(v) < -1, \text{ for all } |v| < 1 \\ \displaystyle\int_{-1}^{\min[\phi(v,1)} (1-u^2)^{(m-4)/2} du, & \text{if} \quad \phi(v) > -1 \\ & \text{for some } v \in (-1,1) \end{cases} \tag{AI.2}$$

where

$$\phi(v) = \frac{(\bar{x} - \bar{y})\sqrt{m}}{S_y(m-1)} + v\,\frac{S_x(n-1)}{S_y(m-1)}\sqrt{\frac{m}{n}} \tag{AI.2a}$$

Then,

$$\hat{R}_2 = \begin{cases} 1 & \text{if} \quad A > 1 \\ \dfrac{\Gamma(\frac{n-1}{2})\Gamma(\frac{m-1}{2})}{\pi\Gamma(\frac{n-2}{2})\Gamma(\frac{m-2}{2})} \displaystyle\int_{-1}^{1} h(v)(1-v^2)^{\frac{n-4}{2}} dv & \text{if} \quad |A| < 1 \\ 0 & \text{if} \quad A < 1 \end{cases} \tag{AI.3}$$

(We note that on page 557 of Downton's aforementioned paper, the last two lines have an error and should have "$\hat{R}_2 = 0$" and "$\hat{R}_2 = 1$" interchanged).

APPENDIX AII

As advertised in Section 3, we wish to derive in this Appendix, the result given in (4.20). We have from (4.19) that

$$1 = \int_{-\infty}^{\infty} p(\delta|\underset{\sim}{x},\underset{\sim}{y})d\delta = \tag{AII.1}$$

$$= K \int_{-\infty}^{\infty}\int_{0}^{\infty} u^{b-1}\exp[-\frac{1}{2}\{n[\delta - (\bar{x} - \bar{y})u]^2 + au^2]dudw$$

where $a > 0$, and $b > 4$. On interchanging the order of integration, we have

$$1 = K \int_{0}^{\infty} u^{b-1}\exp\{-\frac{1}{2} au^2\}\{\int_{-\infty}^{\infty} \exp - \frac{n}{2} [\delta - (\bar{x} - \bar{y})u]^2 d\delta\}du \tag{AII.2}$$

The inner integral of (AII.2) is that of an unnormalized normal density, mean $(\bar{x} - \bar{y})u$, variance $1/n$, so that the inner integral has value $\sqrt{2\pi/n}$.

Hence

$$1 = \frac{\sqrt{2\pi}}{\sqrt{n}} K \int_{0}^{\infty} u^{b-1}\exp\{-\frac{1}{2} au^2\}du \tag{AII.3}$$

If we let $au^2/2 = w$ in the integral, we easily find

$$1 = \frac{\sqrt{2\pi}}{\sqrt{n}} K \frac{2^{(b-2)/2}}{a^{b/2}} \int_{0}^{\infty} w^{\frac{b}{2} - 1} e^{-w}dw \tag{AII.4}$$

so that

$$1 = \frac{K\ 2^{(b-1)/2}\sqrt{\pi}\ \Gamma(\frac{b}{2})}{\sqrt{n}\ a^{b/2}} \tag{AII.4a}$$

or

$$K = \sqrt{n}\ a^{b/2} \Big/ \sqrt{\pi}\ 2^{(b-1)/2}\Gamma(\frac{b}{2}) \tag{AII.5}$$

as stated.

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